Inequality
https://www.linkedin.com/groups/8313943/8313943-6419237830773473284
Let x, y and z are positive real numbers such that
$x^2+y^2+z^2=1$. Prove that
$1/x^2+1/y^2+1/z^2 \geq 3+2(x^3+y^3+z^3)/(xyz)$.
Solution by Arkady Alt , **San Jose**, **California**, **USA**.
Since homogeneous form of original inequality is
$$(x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) \geq 3+2\cdot\frac{x^3+y^3+z^3}{xyz}$$
then using normalization by $x+y+z=1$ and denoting $p:=xy+yz+zx, q:=xyz$
we obtain $\dfrac{(1-2p)(p^2-2q)}{q^2} \geq 3+2\cdot\dfrac{1+3q-3p}{q} \iff$
$(1-2p)(p^2-2q) \geq q(2+9q-6p)$.
Since $3p=3(xy+yz+zx) \leq (x+y+z)^2=1$ then denoting $t:=\sqrt{1-3p}$
we obtain $p=\dfrac{1-t^2}{3}$, where $t \in [0,1)$. Also, since Vieta's system of equations
$x+y+z=1, xy+yz+zx=\dfrac{1-t^2}{3}, xyz=q$ solvable in real x,y,z iff
$\dfrac{(1+t)^2(1-2t)}{27} \leq q \leq \dfrac{(1-t)^2(1+2t)}{27}$ then $(1-2p)(p^2-2q)-q(2+9q-6p) \geq$
$$\left(1-2\cdot\frac{1-t^2}{3}\right)\left(\left(\frac{1-t^2}{3}\right)^2-2\cdot\frac{(1-t)^2(1+2t)}{27}\right)-$$
$$\frac{(1-t)^2(1+2t)}{27}\left(2+9\cdot\frac{(1-t)^2(1+2t)}{27}-6\cdot\frac{1-t^2}{3}\right)=$$
$$\frac{(2t^2+1)(1-t)^2(3t^2+2t+1)}{81}-\frac{(1-t)^2(1+2t)(2t^3+3t^2+1)}{81}=$$
$$=\frac{(1-t)^2}{81}\left((2t^2+1)(3t^2+2t+1)-(1+2t)(2t^3+3t^2+1)\right)=\frac{2}{81}t^2(t-1)^4 \geq 0.$$